ON SOLUTIONS OF KOLMOGOROV'S EQUATIONS FOR JUMP MARKOV PROCESSES

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This paper links three ways to construct a nonhomogeneous jump Markov process: (i) via a compensator of the random measure of a multivariate point process, (ii) as a minimal solution of the backward Kolmogorov equation, and (iii) as a minimal solution of the forward Kolmogorov equation. The main conclusion of this paper is that, for a given measurable Q-function, all these constructions define the same transition function. If this function is a transition probability, it is the unique solution of the backward and forward Kolmogorov equations. For a continuous Q-function, the equality of minimal solutions of the backward and forward Kolmogorov equations and the explicit formulae for these solutions were established by Feller (1940). In particular, this paper extends Feller's (1940) results for continuous Q-functions to measurable Q-functions.

1. Introduction. Let $(X, \mathfrak{B}(X))$ be a standard Borel space, that is, $(X, \mathfrak{B}(X))$ is a measurable space for which there exists a measurable injection onto a Borel subset of the real line endowed with its Borel σ-field. For a Borel subset E of the extended real line, we denote by $\mathfrak{B}(E)$ its Borel σ-field. A function $P(u, x; t, B)$, where $u, t \in \mathbb{R}_+ :=]0, \infty[$, $u < t$, $x \in X$, and $B \in \mathfrak{B}(X)$, is called a transition function if it takes values in $[0, 1]$ and satisfies the following properties:

(i) for all u, x, t the function $P(u, x; t, \cdot)$ is a measure on $(X, \mathfrak{B}(X))$;
(ii) for all B the function $P(u, x; t, B)$ is Borel measurable in (u, x, t);
(iii) $P(u, x; t, B)$ satisfies the Chapman-Kolmogorov equation

$$(1) \qquad P(u, x; t, B) = \int_X P(s, y; t, B) P(u, x; s, dy), \qquad u < s < t.$$

If $P(u, x; t, X) = 1$ for all u, x, t, then the transition function P is called a *transition probability function*.

A stochastic process $\{\mathbb{X}_t : t \geq 0\}$ with values in X, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, is called Markov if $\mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u) = \mathbb{P}(\mathbb{X}_t \in B \mid \mathbb{X}_u)$,

$\mathbb{P} - a.s$ for all $u, t \in \mathbb{R}_+$ with $u < t$ and for all $B \in \mathfrak{B}(X)$. And each Markov process has a transition function P such that $\mathbb{P}(\mathbb{X}_t \in B \mid \mathbb{X}_u) = P(u, \mathbb{X}_u; t, B)$, $\mathbb{P} - a.s$; see Kuznetsov [18], where the equivalence of the two approaches to define a Markov process from Kolmogorov [17] is established. If each sample path of the Markov process is a right continuous piecewise constant function that has a finite number of discontinuity points on each interval $[0, t]$ for $t < t_\infty$, where t_∞ is the lifetime of the process, then the Markov process is called a *jump Markov process*; Gikhman and Skorokhod [8, p. 187]. In this paper, measurability and Borel measurability are used synonymously and all conditional distributions are defined almost sure.

The following function q described below serves as the local characteristic for a Markov process defined by its compensator and also for the backward and forward Kolmogorov equations. A function $q(x, t, B)$, where $x \in X$, $t \in \mathbb{R}_+$, and $B \in \mathfrak{B}(X)$, is called a *Q-function* if it satisfies the following properties:

(i) for all x, t the function $q(x, t, \cdot)$ is a signed measure on $(X, \mathfrak{B}(X))$ such that $q(x, t, X)$ ≤ 0 and $0 \leq q(x, t, B \setminus \{x\}) < \infty$ for all $B \in \mathfrak{B}(X)$;
(ii) for all B the function $q(x, t, B)$ is measurable in (x, t).

In addition to properties (i) and (ii), if $q(x, t, X) = 0$ for all x, t, then the Q-function q is called *conservative*. Note that any Q-function can be transformed into a conservative Q-function by adding a state $\tilde{x}$ to X with $q(x, t, \{\tilde{x}\}) := -q(x, t, X)$, $q(\tilde{x}, t, X) := 0$, and $q(\tilde{x}, t, \{\tilde{x}\}) := 0$, where $x \in X$ and $t \in \mathbb{R}_+$. To simplify the presentation, in this paper we always assume that q is conservative. If there is no assumption that q is conservative, Remark 4.1 explains how the main formulations change. A Q-function q is called *continuous* if it is continuous in $t \in \mathbb{R}_+$.

Let $q(x, t) := -q(x, t, \{x\})$. A set $B \in \mathfrak{B}(X)$ is called *q-bounded* if $\sup_{x \in B, t \in \mathbb{R}_+} q(x, t) < \infty$ and the Q-function q is called *stable* if the set $\{x\}$ is q-bounded for all $x \in X$. We make the following assumption throughout the paper.

ASSUMPTION 1.1.   *The Q-function q is stable.*

Let $B_n := \{x \in X : \sup_{t \in \mathbb{R}_+} q(x, t) < n + 1\}$ for all $n \geq 0$. Then, $B_n \uparrow X$ as $n \to \infty$, if q is stable. Thus, Assumption 1.1 is equivalent to the existence of a sequence of q-bounded sets $\{B_n\}_{n \geq 0}$ such that $B_n \uparrow X$ as $n \to \infty$. This way a stable Q-function was defined in Feller [7].

In this paper, a non-negative solution $\bar{f}$ in a certain class of solutions of a functional equation is called the minimal non-negative solution if for any non-negative solution f of this equation from that class $\bar{f}(\mathbf{x}) \leq f(\mathbf{x})$ for all values of the argument $\mathbf{x}$. Each Q-function defines the backward and forward Kolmogorov equations. For a stable continuous Q-function, Feller [7] found minimal non-negative solutions of the backward and forward Kolmogorov equations, showed that these minimal solutions are equal, and that the minimal solution is a transition function. For homogeneous Markov processes, that is Q-functions

do not depend on t, Doob [3, 4, Chap. 6] provided an explicit construction for multiple transition functions satisfying the backward Kolmogorov equation, Kendall [13], Kendall and Reuter [14], and Reuter [19] gave examples with non-unique solutions to Kolmogorov equations, and Reuter [19] provided necessary and sufficient conditions for their uniqueness; see also Chen et al. [2]. Ye, Guo, and Hernández-Lerma [20] investigated a countable state problem with Borel measurable Q-functions.

A conservative Q-function can be used to construct a predictable random measure. According to Jacod [12, Theorem 3.6], an initial state distribution and a predictable random measure define uniquely a multivariate point process. Theorem 2.2 given below states that the stochastic process associated with the multivariate point process defined by a stable conservative Q function q and an initial state distribution is a jump Markov process. Theorem 2.2 also describes its transition function $\bar{P}$. By using the methods introduced by Feller [7] for continuous Q-functions, in Theorems 3.1, 4.1 we show that the transition function $\bar{P}$ satisfies the backward and forward Kolmogorov equations written for the Q-function q. In addition, $\bar{P}$ is the minimal non-negative solution of the backward and forward Kolmogorov equations and, if this transition function is a transition probability, then it is the unique non-negative solution of the backward and forward Kolmogorov equations that is a measure with values in $[0, 1]$; Theorems 3.2, 4.3. In particular, this paper extends Feller's [7] results from continuous to measurable Q-functions. Though the uniqueness property of the transition probability function $\bar{P}$ was stated in Feller [7, Erratum], the proof is missing.

Our interest in this study is motivated by its applications to continuous-time Markov decision processes. Here we mention two of them:

(i) For a countable state space, each Markov policy along with a given initial state distribution defines a Markov process with the transition function being the minimal solution of the forward Kolmogorov equation; Guo and Hernández-Lerma [9, Section 2.2]. An arbitrary policy defines a multivariate point process via the compensator of its random measure; Kitaev [15], Kitaev and Rykov [16, Section 4.6], Feinberg [5, 6], Guo and Piunovskiy [10]. The results of this paper imply that for Markov policies these two definitions are equivalent for problems with Borel state spaces.

(ii) Feller's [7] results are broadly used in the literature on continuous-time Markov decision processes to define Markov processes corresponding to Markov policies; see Guo and Hernández-Lerma [9] and references therein. The application of Feller's results typically leads to the unnecessary assumption that decisions depend continuously on time; see, e.g., [11, Definition 2.2]. For countable state problems the results in Ye et al. [20] removed the necessity to assume this continuity. The results of the current paper imply that this continuity assumption is unnecessary for Markov decision processes with Borel state spaces.

2. Relation between Jump Markov Processes and Q-functions. The main goals of this section are to show that an initial state distribution and a (stable) Q-function q define a jump Markov process and to construct its transition function.

Let $x_\infty \notin X$ be an isolated point adjoined to the space X. Denote $\bar{X} = X \cup \{x_\infty\}$ and $\bar{\mathbb{R}}_+ =]0,\infty]$. Consider the Borel σ-field $\mathfrak{B}(\bar{X}) = \sigma(\mathfrak{B}(X), \{x_\infty\})$ on $\bar{X}$, which is the minimal σ-field containing $\mathfrak{B}(X)$ and $\{x_\infty\}$. Let $(\bar{X} \times \bar{\mathbb{R}}_+)^\infty$ be the set of all sequences $(x_0, t_1, x_1, t_2, x_2, \ldots)$ with $x_n \in \bar{X}$ and $t_{n+1} \in \bar{\mathbb{R}}_+$ for all $n \geq 0$. This set is endowed with the σ-field generated by the products of the Borel σ-fields $\mathfrak{B}(\bar{X})$ and $\mathfrak{B}(\bar{\mathbb{R}}_+)$.

Denote by Ω the subset of all sequences $\omega = (x_0, t_1, x_1, t_2, x_2, \ldots)$ from $(\bar{X} \times \bar{\mathbb{R}}_+)^\infty$ such that: (i) $x_0 \in X$; (ii) if $t_n < \infty$, then $t_n < t_{n+1}$ and $x_n \in X$, and, if $t_n = \infty$, then $t_{n+1} = t_n$ and $x_n = x_\infty$, for all $n \geq 1$. Observe that Ω is a measurable subset of $(\bar{X} \times \bar{\mathbb{R}}_+)^\infty$. Consider the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the σ-field of the measurable subsets of Ω. Then, $x_n(\omega) = x_n$ and $t_{n+1}(\omega) = t_{n+1}$, $n \geq 0$, are random variables defined on the measurable space $(\Omega, \mathcal{F})$. Let $t_0 := 0$, $t_\infty(\omega) := \lim_{n\to\infty} t_n(\omega)$, $\omega \in \Omega$, and for all $t \geq 0$ let $\mathcal{F}_t := \sigma(\mathfrak{B}(X), \mathcal{G}_t)$, where $\mathcal{G}_t := \sigma(I\{x_n \in B\}I\{t_n \leq s\} : n \geq 1, 0 \leq s \leq t, B \in \mathfrak{B}(X))$. Throughout this paper, we omit ω whenever possible and also follow the standard convention that $0 \times \infty = 0$.

For a given Q-function q, consider the random measure ν on $(\mathbb{R}_+ \times X, \mathfrak{B}(\mathbb{R}_+) \times \mathfrak{B}(X))$ defined by

$$(2) \quad \nu(\omega;]0,t], B) = \int_0^t \sum_{n\geq 0} I\{t_n < s \leq t_{n+1}\} q(x_n, s, B \setminus \{x_n\}) ds, \quad t \in \mathbb{R}_+, \ B \in \mathfrak{B}(X).$$

Note that $\nu(\{t\}, X) = \nu([t_\infty, \infty[, X) = 0$ and (2) can be rearranged as

$$(3) \quad \nu(]0,t], B) = \sum_{n\geq 0} I\{t_n < t \leq t_{n+1}\} \left(\sum_{m=0}^{n-1} \int_0^{t_{m+1}-t_m} q(x_m, t_m + s, B \setminus \{x_m\}) ds \right.$$
$$\left. + \int_0^{t-t_n} q(x_n, t_n + s, B \setminus \{x_n\}) ds \right).$$

As the expression in the parentheses on the right hand side of (3) is an $\mathcal{F}_{t_n}$-measurable process for each $B \in \mathfrak{B}(X)$, it follows from Jacod [12, Lemma 3.3] that the process $\{\nu(]0,t], B) : t \in \mathbb{R}_+\}$ is predictable. Therefore, the measure ν is a predictable random measure. According to Jacod [12, Theorem 3.6], the predictable random measure ν defined in (2) and a probability measure μ on X define a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(x_0 \in B) = \mu(B)$, $B \in \mathfrak{B}(X)$, and ν is the compensator of the random measure of the multivariate point process $(t_n, x_n)_{n\geq 1}$ defined by the triplet $(\Omega, \mathcal{F}, \mathbb{P})$.

Consider the process $\{\mathbb{X}_t : t \geq 0\}$,

$$(4) \qquad\qquad \mathbb{X}_t(\omega) := \sum_{n\geq 0} I\{t_n \leq t < t_{n+1}\} x_n + I\{t_\infty \leq t\} x_\infty,$$

defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We abbreviate the process $\{\mathbb{X}_t : t \geq 0\}$ as $\mathbb{X}$. The main result of this section, Theorem 2.2, shows that the process $\mathbb{X}$ is a jump Markov process and provides its transition function.

For an $\mathcal{F}_t$-measurable stopping time τ, let $N(\tau) := \max\{n = 0, 1, \ldots : \tau \geq t_n\}$. Since $N(\tau) = \infty$ and $\mathbb{X}_\tau = \{x_\infty\}$ when $\tau \geq t_\infty$, we follow the convention that $t_{\infty+1} = \infty$ and $x_{\infty+1} = x_\infty$. Denote by $G_\tau(\omega; \cdot, \cdot)$ and $H_\tau(\omega; \cdot)$ respectively the regular conditional laws of $(t_{N(\tau)+1}, x_{N(\tau)+1})$ and $t_{N(\tau)+1}$ with respect to $\mathcal{F}_\tau$; $H_\tau(\omega; \cdot) = G_\tau(\omega; \cdot, \bar{X})$. In particular, $G_{t_n}(\omega; \cdot, \cdot)$ and $H_{t_n}(\omega; \cdot)$, where $n = 0, 1, \ldots$, denote the conditional laws of (t_{n+1}, x_{n+1}) and t_{n+1} with respect to $\mathcal{F}_{t_n}$. We remark that the notations G_{t_n} and H_{t_n} correspond to the notations G_n and H_n in Jacod [12, p. 241].

LEMMA 2.1.   *For all $u, t \in \mathbb{R}_+$, $u < t$,*

$$(5) \qquad H_u([t, \infty]) = e^{-\int_u^t q(\mathbb{X}_u, s) ds}, \qquad\qquad\qquad N(u) < \infty,$$

$$(6) \qquad G_u(dt, B) = e^{-\int_u^t q(\mathbb{X}_u, s) ds} q(\mathbb{X}_u, t, B \setminus \{\mathbb{X}_u\}) dt, \qquad B \in \mathfrak{B}(X), \; N(u) < \infty.$$

PROOF. According to Jacod [12, Proposition 3.1], for all $t \in \mathbb{R}_+$, $B \in \mathfrak{B}(X)$, and $n = 0, 1, \ldots$

$$(7) \qquad\qquad \nu(dt, B) = \frac{G_{t_n}(dt, B)}{H_{t_n}([t, \infty])}, \qquad t_n < t \leq t_{n+1}.$$

In particular, for $B = X$, from (7) and from the property that $x_{n+1} \in X$ when $t_{n+1} < \infty$,

$$\nu(dt, X) = \frac{G_{t_n}(dt, X)}{H_{t_n}([t, \infty])} = \frac{H_{t_n}(dt)}{H_{t_n}([t, \infty])}, \qquad t_n < t \leq t_{n+1}.$$

This equality implies that $\nu(dt, X)$ is the hazard rate function corresponding to the distribution H_{t_n} when $t_n < t \leq t_{n+1}$. Therefore,

$$(8) \qquad\qquad H_{t_n}([t, \infty]) = e^{-\nu(]t_n, t], X) I\{t_n < t \leq t_{n+1}\}}, \qquad t \in \mathbb{R}_+, t > t_n.$$

From (2) and (8), for all $t \in \mathbb{R}_+$,

$$(9) \qquad\qquad H_{t_n}([t, \infty]) = e^{-\int_{t_n}^t q(x_n, s) ds}, \qquad t > t_n,$$

and from (2), (7), and (9), for all $t \in \mathbb{R}_+$, $B \in \mathfrak{B}(X)$,

$$(10) \qquad\qquad G_{t_n}(dt, B) = e^{-\int_{t_n}^t q(x_n, s) ds} q(x_n, t, B \setminus \{x_n\}) dt, \qquad t > t_n.$$

To compute G_u, observe that for all $u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$,

$$(11) \quad G_u(dt, B) = \mathbb{P}(t_{N(u)+1} \in [t, t+dt[, x_{N(u)+1} \in B \mid \mathcal{F}_u)$$
$$= \sum_{n \geq 0} \mathbb{P}(t_{N(u)+1} \in [t, t+dt[, x_{N(u)+1} \in B \mid \mathcal{F}_u) I\{N(u) = n\}$$
$$= \sum_{n \geq 0} \mathbb{P}(t_{n+1} \in [t, t+dt[, x_{n+1} \in B \mid \mathcal{F}_u, N(u) = n) I\{N(u) = n\},$$

where the first equality follows from the definition of G_u, the second equality holds because $\{N(u) = \infty\} \cup \{N(u) = n\}_{n=0,1,\dots}$ is an $\mathcal{F}_u$-measurable partition of Ω and $x_{N(u)+1} = x_\infty \notin X$ when $N(u) = \infty$, and the third equality follows from $N(u) = n$ and from $\{N(u) = n\} \in \mathcal{F}_u$. Observe that for any random variable Z on $(\Omega, \mathcal{F})$

$$
\begin{aligned}
(12) \quad \mathbb{P}(Z \mid \mathcal{F}_u, N(u) = n)I\{N(u) = n\} &= \mathbb{P}(Z \mid \mathcal{F}_{t_n}, N(u) = n)I\{N(u) = n\} \\
&= \mathbb{P}(Z \mid \mathcal{F}_{t_n}, t_n \le u, t_{n+1} > u)I\{N(u) = n\} = \mathbb{P}(Z \mid \mathcal{F}_{t_n}, t_{n+1} > u)I\{N(u) = n\} \\
&= \frac{\mathbb{P}(Z, t_{n+1} > u \mid \mathcal{F}_{t_n})}{\mathbb{P}(t_{n+1} > u \mid \mathcal{F}_{t_n})}I\{N(u) = n\},
\end{aligned}
$$

where the first equality follows from Brémaud [1, Theorem T32, p. 308], the second equality holds because $\{t_n \le u, t_{n+1} > u\} = \{N(u) = n\}$, the third equality holds because $\{t_n \le u\} \in \mathcal{F}_{t_n}$, and the last one follows from the definition of conditional probabilities.

Let $Z = \{t_{n+1} \in [t, t + dt[\,, x_{n+1} \in B\}$, where $t \in \mathbb{R}_+, B \in \mathfrak{B}(X)$. Then (11) and (12) imply

$$
\begin{aligned}
(13) \quad G_u(dt, B) &= \sum_{n \ge 0} \frac{\mathbb{P}(t_{n+1} \in [t, t + dt[\,, x_{n+1} \in B \mid \mathcal{F}_{t_n})}{\mathbb{P}(t_{n+1} > u \mid \mathcal{F}_{t_n})}I\{N(u) = n\} \\
&= \sum_{n \ge 0} \frac{e^{-\int_{t_n}^{t} q(x_n, s)ds} q(x_n, t, B \setminus \{x_n\})dt}{e^{-\int_{t_n}^{u} q(x_n, s)ds}}I\{N(u) = n\} \\
&= e^{-\int_u^t q(\mathbb{X}_u, s)ds} q(\mathbb{X}_u, t, B \setminus \{\mathbb{X}_u\})dt,
\end{aligned}
$$

where the first equality holds because $\{t_{n+1} \in [t, t + dt[\,, t_{n+1} > u\} = \{t_{n+1} \in [t, t + dt[\}$ when $t > u$, the second equality follows from (9) and (10), and the last equality holds since $x_n = \mathbb{X}_u$ when $N(u) = n$. For all $u, t \in \mathbb{R}_+, u < t$, it follows from the property that $x_{N(u)+1} \in X$ when $t_{N(u)+1} < \infty$ and from (13) that $H_u([t, \infty])$ satisfies (5). $\qquad \square$

Following Feller [7, p. 501], for $x \in X$, $u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$, define

$$
(14) \qquad\qquad \bar{P}^{(0)}(u, x; t, B) = I\{x \in B\}e^{-\int_u^t q(x, s)ds},
$$

and for $n \ge 1$ define

$$
(15) \qquad \bar{P}^{(n)}(u, x; t, B) = \int_u^t \int_{X \setminus \{x\}} e^{-\int_u^s q(x, \theta)d\theta} q(x, s, dy)\bar{P}^{(n-1)}(s, y; t, B)ds.
$$

Set

$$
(16) \qquad\qquad \bar{P}(u, x; t, B) := \sum_{n=0}^{\infty} \bar{P}^{(n)}(u, x; t, B).
$$

Observe that $\bar{P}$ is a transition function. For stable continuous Q-functions, Feller [7, Theorems 2, 5] proved that (a) for fixed u, x, t the function $\bar{P}(u, x; t, \cdot)$ is a measure on $(X, \mathfrak{B}(X))$ such that $0 \leq \bar{P}(u, x; t, \cdot) \leq 1$, and (b) for all u, x, t, B the function $\bar{P}(u, x; t, B)$ satisfies the Chapman-Kolmogorov equation (1). The proofs remain correct for measurable Q-functions q. The measurability of $\bar{P}(u, x; t, B)$ in u, x, t for all $B \in \mathfrak{B}(X)$ is straightforward from the definitions (14), (15), and (16). Therefore, the function $\bar{P}$ satisfies properties (i)-(iii) from the definition of a transition function.

THEOREM 2.2. *For a given initial state distribution and for a stable Q-function q, the process $\mathbb{X}$ defined in (4) is a jump Markov process with the transition function $\bar{P}$.*

PROOF. Observe that the sample paths of the process $\mathbb{X}$ are right-continuous piecewise-constant functions that have finite number of discontinuities on each interval $[0, t]$ for $t < t_\infty$. Thus if, for all $u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$,

$$(17) \qquad \mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u) = \mathbb{P}(\mathbb{X}_t \in B \mid \mathbb{X}_u) = \bar{P}(u, \mathbb{X}_u; t, B), \qquad u < t_\infty,$$

then the process $\mathbb{X}$ is a jump Markov process with the transition function $\bar{P}$. To prove (17), we first establish by induction that for all $n = 0, 1, \dots$, $u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$

$$(18) \qquad \mathbb{P}(\mathbb{X}_t \in B, N_{]u,t]} = n \mid \mathcal{F}_u) = \bar{P}^{(n)}(u, \mathbb{X}_u; t, B), \qquad u < t_\infty,$$

where $N_{]u,t]} := N(t) - N(u)$ when $u < t_\infty$ and $N_{]u,t]} := \infty$ when $u \geq t_\infty$. Equation (18) holds for $n = 0$ because for $u < t_\infty$

$$(19) \quad \mathbb{P}(\mathbb{X}_t \in B, N_{]u,t]} = 0 \mid \mathcal{F}_u) = \mathbb{P}(\mathbb{X}_u \in B, t_{N(u)+1} > t \mid \mathcal{F}_u)$$
$$= I\{\mathbb{X}_u \in B\} H_u(]t, \infty]) = I\{\mathbb{X}_u \in B\} e^{-\int_u^t q(\mathbb{X}_u, s) ds} = \bar{P}^{(0)}(u, \mathbb{X}_u; t, B),$$

where the first equality holds because the corresponding events coincide, the second equality holds because $\{\mathbb{X}_u \in B\} \in \mathcal{F}_u$ and from the definition of H_u, the third equality is correct because of (5), and the last equality is (14).

For some $n \geq 0$, assume that (18) holds. Then for $u < t_\infty$

$$(20) \quad \mathbb{P}(\mathbb{X}_t \in B, N_{]u,t]} = n + 1 \mid \mathcal{F}_u)$$
$$= \int_u^t \int_{X \setminus \{\mathbb{X}_u\}} \mathbb{P}(\mathbb{X}_t \in B, N_{]t_{N(u)+1}, t]} = n \mid \mathcal{F}_u, t_{N(u)+1}, x_{N(u)+1}) G_u(dt_{N(u)+1}, dx_{N(u)+1})$$
$$= \int_u^t \int_{X \setminus \{\mathbb{X}_u\}} \mathbb{P}(\mathbb{X}_t \in B, N_{]t_{N(u)+1}, t]} = n \mid \mathcal{F}_{t_{N(u)+1}}) G_u(dt_{N(u)+1}, dx_{N(u)+1})$$
$$= \int_u^t \int_{X \setminus \{\mathbb{X}_u\}} q(\mathbb{X}_u, s, dy) e^{-\int_u^s q(\mathbb{X}_u, \theta) d\theta} \bar{P}^{(n)}(s, y; t, B) ds = \bar{P}^{(n+1)}(u, \mathbb{X}_u; t, B),$$

where the first equality holds since $N_{]u,t]} = 1 + N_{]t_{N(u)+1},t]}$ for $N_{]u,t]} \geq 1$ and since $\mathbb{E}(\mathbb{E}(Z \mid \mathfrak{D})) = \mathbb{E}(Z)$ for any random variable Z and any σ-field $\mathfrak{D}$, the second equality holds since $\sigma(\mathcal{F}_u, t_{N(u)+1}, x_{N(u)+1}) = \mathcal{F}_{t_{N(u)+1}}$, the third equality follows from (6) and (18), and the last equality is (15). Equality (18) is proved.

Observe that for $u, t \in \mathbb{R}_+$, $u < t$, $B \in \mathfrak{B}(X)$,

$$
\begin{aligned}
(21) \quad \mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u) &= \mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u) I\{u < t_\infty\} + \mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u) I\{u \geq t_\infty\} \\
&= \sum_{n \geq 0} \mathbb{P}(\mathbb{X}_t \in B, N_{]u,t]} = n \mid \mathcal{F}_u) I\{u < t_\infty\} = \sum_{n \geq 0} \bar{P}^{(n)}(u, \mathbb{X}_u; t, B) I\{u < t_\infty\} \\
&= \bar{P}(u, \mathbb{X}_u; t, B) I\{u < t_\infty\} = \bar{P}(u, \mathbb{X}_u; t, B) I\{\mathbb{X}_u \in X\},
\end{aligned}
$$

where the first equality holds since $\{\{u < t_\infty\}, \{u \geq t_\infty\}\}$ is a partition of Ω and $\{u < t_\infty\}, \{u \geq t_\infty\} \in \mathcal{F}_u$, the second equality holds since $\mathbb{X}_t \in X$ implies $t < t_\infty$, the third equality follows from (18), the fourth equality follows from (16), and the last one holds since $\{u < t_\infty\} = \{\mathbb{X}_u \in X\}$. As follows from (21), the function $\mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u)$ is $\sigma(\mathbb{X}_u)$-measurable. Thus,

$$
(22) \qquad \mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u) = \mathbb{P}(\mathbb{P}(\mathbb{X}_t \in B \mid \mathcal{F}_u) \mid \mathbb{X}_u) = \mathbb{P}(\mathbb{X}_t \in B \mid \mathbb{X}_u),
$$

where the second equality holds because $\sigma(\mathbb{X}_u) \subseteq \mathcal{F}_u$; see e.g. Brémaud [1, p. 280]. Thus, (17) follows from (21) and (22). $\qquad \square$

3. Backward Kolmogorov equation. In this section, we show that the transition function $\bar{P}$ defined in (16) is the minimal non-negative solution to the backward Kolmogorov equation. For a continuous Q-function q, relevant results were established by Feller [7, Theorems 2, 3].

THEOREM 3.1.   *The function $\bar{P}(u, x; t, B)$ satisfies the following properties:*
(i) $\bar{P}(u, x; t, B)$ is for fixed x, t, B an absolutely continuous function in u and satisfies uniformly in $B \in \mathfrak{B}(X)$ the boundary condition

$$
(23) \qquad \lim_{u \to t^-} \bar{P}(u, x; t, B) = I\{x \in B\}.
$$

(ii) For all x, t, B, the function $\bar{P}(u, x; t, B)$ satisfies for almost every $u < t$ the backward Kolmogorov equation

$$
(24) \qquad \frac{\partial}{\partial u} P(u, x; t, B) = q(x, u) P(u, x; t, B) - \int_{X \setminus \{x\}} q(x, u, dy) P(u, y; t, B).
$$

PROOF. (i) For all $x \in X$, $u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$,

$$(25) \quad \bar{P}(u, x; t, B) = \sum_{n=0}^{\infty} \bar{P}^{(n)}(u, x; t, B)$$

$$= I\{x \in B\}e^{-\int_u^t q(x,s)ds} + \sum_{n=1}^{\infty} \int_u^t e^{-\int_u^s q(x,\theta)d\theta} \int_{X\setminus\{x\}} q(x, s, dy)\bar{P}^{(n-1)}(s, y; t, B)ds$$

$$= I\{x \in B\}e^{-\int_u^t q(x,s)ds} + \int_u^t e^{-\int_u^s q(x,\theta)d\theta} \int_{X\setminus\{x\}} q(x, s, dy)\sum_{n=1}^{\infty} \bar{P}^{(n-1)}(s, y; t, B)ds$$

$$= I\{x \in B\}e^{-\int_u^t q(x,s)ds} + \int_u^t e^{-\int_u^s q(x,\theta)d\theta} \int_{X\setminus\{x\}} q(x, s, dy)\bar{P}(s, y; t, B)ds,$$

where the first equality is (16), the second equality follows from (14) and (15), the third equality is obtained by interchanging the integral and sum, and the last one follows from (16). For fixed x, t, B, equation (25) implies that $\bar{P}(u, x; t, B)$ is the sum of two absolutely continuous functions in u. Thus, $\bar{P}(u, x; t, B)$ is for fixed x, t, B an absolutely continuous function in u.

Observe that $\bar{P}^{(n)}(u, x; t, B) \le \bar{P}(u, x; t, B) \le 1$ for all $n \ge 0$, $x \in X, u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$. Then from (15),

$$(26) \qquad \bar{P}^{(n)}(u, x; t, B) \le \int_u^t e^{-\int_u^s q(x,\sigma)d\sigma} q(x, s)ds, \qquad n \ge 1.$$

This inequality and (14) imply that, for any stable Q-function q,

$$(27) \quad \lim_{u \to t^-} \bar{P}^{(n)}(u, x; t, B) = 0 \quad \text{for all } n \ge 1 \quad \text{and} \quad \lim_{u \to t^-} \bar{P}^{(0)}(u, x; t, B) = I\{x \in B\}$$

uniformly with respect to B. Thus, (16) and (27) imply (23).

(ii) Since an absolutely continuous real-valued function is differentiable almost everywhere on its domain, for all x, t, B the function $\bar{P}(u, x; t, B)$ is differentiable in u almost everywhere on $]0, t[$. By differentiating (25), for almost every $u < t$,

$$(28) \quad \frac{\partial}{\partial u}\bar{P}(u, x; t, B) = I\{x \in B\}e^{-\int_u^t q(x,s)ds}q(x, u) - \int_{X\setminus\{x\}} q(x, u, dy)\bar{P}(u, y; t, B)$$

$$+ \int_u^t \frac{\partial}{\partial u}e^{-\int_u^s q(x,\theta)d\theta} \int_{X\setminus\{x\}} q(x, s, dy)\bar{P}(s, y; t, B)ds$$

$$= I\{x \in B\}e^{-\int_u^t q(x,s)ds}q(x, u) - \int_{X\setminus\{x\}} q(x, u, dy)\bar{P}(u, y; t, B)$$

$$+ \int_u^t e^{-\int_u^s q(x,\theta)d\theta}q(x, u) \int_{X\setminus\{x\}} q(x, s, dy)\bar{P}(s, y; t, B)ds.$$

In view of (25), the sum of the first and the last terms in the last expression of (28) is equal to the first term on the right-hand side of (24). $\qquad\square$

As shown in Feller [7, Theorem 2], for a stable continuous Q-function q, the transition function $\bar{P}$ satisfies the backward Kolmogorov equation for all u, while Theorem 3.1(ii) states that this equation holds for almost every u. This difference in formulations takes place because the continuity of the Q-function q and the finiteness of each integrand in the last expression of (25) guarantee the existence of the derivative $\frac{\partial}{\partial u}\bar{P}(u,x;t,B)$ for all u.

DEFINITION 3.1.   *A function P with the same domain as $\bar{P}$ is a solution of the backward Kolmogorov equation* (24) *if the function P satisfies the properties stated in Theorem 3.1.*

The next theorem describes the minimal and uniqueness properties of the solution $\bar{P}$ of the backward Kolmogorov equation (24).

THEOREM 3.2.   *The function $\bar{P}$ is the minimal non-negative solution of the backward Kolmogorov equation* (24). *Also, if $\bar{P}$ is a transition probability function (that is, $\bar{P}(u,x;t,X) = 1$ for all u,x,t in the domain of $\bar{P}$), then $\bar{P}$ is the unique non-negative solution of the backward Kolmogorov equation* (24) *that is a measure on $(X, \mathfrak{B}(X))$ for fixed u,x,t with $u < t$ and takes values in $[0,1]$.*

PROOF. The proof of minimality is similar to the proof of Theorem 3 in Feller [7]. We provide it here for completeness. Let P^* with the same domain as $\bar{P}$ be a non-negative solution of the backward Kolmogorov equation (24). Integrating (24) from u to t and by using the boundary condition (23),

$$(29) \quad P^*(u,x;t,B) = I\{x \in B\}e^{-\int_u^t q(x,s)ds}$$
$$+ \int_u^t \int_{X\setminus\{x\}} e^{-\int_u^s q(x,\theta)d\theta} q(x,s,dy)P^*(s,y;t,B)ds.$$

Since the last term of (29) is non-negative,

$$(30) \qquad\qquad P^*(u,x;t,B) \geq I\{x \in B\}e^{-\int_u^t q(x,s)ds} = \bar{P}^{(0)}(u,x;t,B),$$

where the last equality is (14). For all u,x,t,B with $u < t$, assume $P^*(u,x;t,B) \geq \sum_{m=0}^{n} \bar{P}^{(m)}(u,x;t,B)$ for some $n \geq 0$. Then from (29)

$$P^*(u,x;t,B) \geq I\{x \in B\}e^{-\int_u^t q(x,s)ds}$$
$$+ \int_u^t \int_{X\setminus\{x\}} e^{-\int_u^s q(x,\theta)d\theta} q(x,s,dy) \sum_{m=0}^{n} \bar{P}^{(m)}(s,y;t,B)ds$$
$$= \bar{P}^{(0)}(u,x;t,B) + \sum_{m=0}^{n} \bar{P}^{(m+1)}(u,x;t,B) = \sum_{m=0}^{n+1} \bar{P}^{(m)}(u,x;t,B),$$

where the first equality follows from the assumption that $P^*(u, x; t, B) \geq \sum_{m=0}^{n} \bar{P}^{(m)}(u, x; t, B)$
for all u, x, t, B with $u < t$, the second equality follows from (14) and (15), and the third
equality is straightforward. Thus, by induction, $P^*(u, x; t, B) \geq \sum_{m=0}^{n} \bar{P}^{(m)}(u, x; t, B)$ for all
$n \geq 0$, $x \in X$, $u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$, which implies that $P^*(u, x; t, B) \geq \bar{P}(u, x; t, B)$ for all u, x, t, B.

To prove the second part of the theorem, let the solution P^* be a measure on $(X, \mathfrak{B}(X))$
for fixed u, x, t and with values in $[0, 1]$. Assume that $P^*(u, x; t, B) \neq \bar{P}(u, x; t, B)$ for at
least one tuple (u, x, t, B). Then,

$$\begin{aligned} P^*(u, x; t, X) &= P^*(u, x; t, B) + P^*(u, x; t, B^c) \\ &> \bar{P}(u, x; t, B) + \bar{P}(u, x; t, B^c) = \bar{P}(u, x; t, X) = 1, \end{aligned}$$

where the inequality holds because $P^*(u, x, t, \cdot) \geq \bar{P}(u, x, t, \cdot)$ for all u, x, t. Since P^* takes
values in $[0, 1]$, the assumption that $P^*(u, x; t, B) \neq \bar{P}(u, x; t, B)$ for atleast one tuple
(u, x, t, B) leads to a contradiction. $\qquad\square$

4. Forward Kolmogorov equation. For the forward Kolmogorov equation, this section provides the results similar to the results on backward Kolmogorov equation in Section 3.

THEOREM 4.1.   *The function* $\bar{P}(u, x; t, B)$ *satisfies the following properties:*
(i) $\bar{P}(u, x; t, B)$ *is for fixed* u, x, B *an absolutely continuous function in* t *and satisfies
uniformly in* $B \in \mathfrak{B}(X)$ *the boundary condition*

$$\text{(31)} \qquad\qquad \lim_{t \to u^+} \bar{P}(u, x; t, B) = I\{x \in B\}.$$

(ii) For all $u, x,$ *and* q*-bounded sets* B*, the function* $\bar{P}(u, x; t, B)$ *satisfies for almost every*
$t > u$ *the forward Kolmogorov equation*

$$\text{(32)} \qquad \frac{\partial}{\partial t} P(u, x; t, B) = -\int_B q(y, t) P(u, x; t, dy) + \int_X q(y, t, B \setminus \{y\}) P(u, x; t, dy).$$

PROOF. (i) For all $x \in X, u, t \in \mathbb{R}_+$, $u < t$, and $B \in \mathfrak{B}(X)$, equation (25) implies that
the function $\bar{P}(u, x; t, B)$ is absolutely continuous in t for fixed u,x,B. Also, equations (14)
and (26) imply that, for any stable Q-function q,

$$\text{(33)} \quad \lim_{t \to u^+} \bar{P}^{(n)}(u, x; t, B) = 0 \quad \text{for all } n \geq 1 \quad \text{and} \quad \lim_{t \to u^+} \bar{P}^{(0)}(u, x; t, B) = I\{x \in B\}$$

uniformly with respect to B. Thus, (16) and (33) imply (31).

(ii) Consider the following non-negative function defined on the domain of $\bar{P}$ by

$$(34) \qquad \Pi(u, x; t, B) = \int_{B \setminus \{x\}} q(x, u, dy) e^{- \int_u^t q(y,s)ds}.$$

According to Feller [7, Theorem 4], the function $\bar{P}^{(n)}(u, x; t, B)$, $n \geq 1$, satisfies the recursion

$$(35) \qquad \bar{P}^{(n)}(u, x; t, B) = \int_u^t \int_X \Pi(s, y; t, B) \bar{P}^{(n-1)}(u, x; s, dy) ds.$$

Though the function $\bar{P}^{(n)}(u, x; t, B)$ is defined for continuous Q-functions in Feller [7], the proof given there is correct for Borel Q-functions. From (14), (16), and (35),

$$
\begin{aligned}
(36) \quad \bar{P}(u, x; t, B) &= \sum_{n=0}^{\infty} \bar{P}^{(n)}(u, x; t, B) \\
&= I\{x \in B\} e^{- \int_u^t q(x,s)ds} + \sum_{n=1}^{\infty} \int_u^t \int_X \Pi(s, y; t, B) \bar{P}^{(n-1)}(u, x; s, dy) ds \\
&= I\{x \in B\} e^{- \int_u^t q(x,s)ds} + \int_u^t \int_X \Pi(s, y; t, B) \sum_{n=1}^{\infty} \bar{P}^{(n-1)}(u, x; s, dy) ds \\
&= I\{x \in B\} e^{- \int_u^t q(x,s)ds} + \int_u^t \int_X \Pi(s, y; t, B) \bar{P}(u, x; s, dy) ds.
\end{aligned}
$$

Since $\bar{P}(u, x; t, B)$ is an absolutely continuous function in t for fixed u, x, B, the derivative $\frac{\partial}{\partial t} \bar{P}(u, x; t, B)$ exists for almost every $t \in]u, \infty[$. By differentiating (36), for almost every $t > u$,

$$
\begin{aligned}
(37) \quad \frac{\partial}{\partial t} \bar{P}(u, x; t, B) = &-I\{x \in B\} e^{- \int_u^t q(x,s)ds} q(x, t) \\
&+ \int_X \Pi(t, y; t, B) \bar{P}(u, x; t, dy) + \int_u^t \frac{\partial}{\partial t} \int_X \Pi(s, y; t, B) \bar{P}(u, x; s, dy) ds
\end{aligned}
$$

By differentiating (34) with respect to t, for all q-bounded sets $B \in \mathcal{B}(X)$,

$$(38) \quad \frac{\partial}{\partial t} \Pi(u, x; t, B) = \int_{B \setminus \{x\}} q(x, u, dy) \frac{\partial}{\partial t} e^{- \int_u^t q(y,s)ds} = - \int_B q(y, t) \Pi(u, x; t, dy).$$

Combining (37) and (38) and observing that $\Pi(t, y; t, B) = q(y, t, B \setminus \{y\})$, for all q-bounded sets B,

$$(39) \quad \frac{\partial}{\partial t} \bar{P}(u, x; t, B) = -I\{x \in B\} e^{- \int_u^t q(x,s)ds} q(x, t)$$

$$+ \int_X q(y, t, B \setminus \{y\}) \bar{P}(u, x; t, dy) - \int_u^t \int_X \int_B q(z, t) \Pi(s, y; t, dz) \bar{P}(u, x; s, dy) ds,$$

for almost every $t > u$. By substituting $\bar{P}(u, x; t, dz)$ in the left-hand side of the following equality with the final expression in (36),

(40)
$$\int_B q(z, t) \bar{P}(u, x; t, dz) = I\{x \in B\} e^{-\int_u^t q(x, s) ds} q(x, t)$$

$$+ \int_u^t \int_X \int_B q(z, t) \Pi(s, y; t, dz) \bar{P}(u, x; s, dy) ds.$$

Formulae (39) and (40) imply statement (ii) of the theorem. $\qquad \square$

COROLLARY 4.2 (Feller [7, Equation (37)]). *For all $x \in X, u, t \in \mathbb{R}_+$, $u < t$, and q-bounded sets $B \in \mathfrak{B}(X)$, the function $\bar{P}(u, x; t, B)$ defined in (16) satisfies*

(41) $\quad P(u, x; t, B) = I\{x \in B\}$
$$+ \int_u^t ds \int_X q(y, s, B \setminus \{y\}) P(u, x; s, dy) - \int_u^t ds \int_B q(y, s) P(u, x; s, dy).$$

PROOF. Integrating (32) from u to t and by using the boundary condition (31), we get (41). Thus, by Theorem 4.1, for all $x \in X, u, t \in \mathbb{R}_+$, $u < t$, and q-bounded sets $B \in \mathfrak{B}(X)$, the function $\bar{P}(u, x; t, B)$ satisfies (41). $\qquad \square$

DEFINITION 4.1. *A function P with the same domain as $\bar{P}$ is a solution of the forward Kolmogorov equation (32) if the function P satisfies the properties stated in Theorem 4.1.*

Following the proof of Theorem 3.2, we establish the minimal and uniqueness properties of the solution $\bar{P}$ of the forward Kolmogorov equation (32) in theorem 4.3. We remark that the function $\bar{P}$ is the minimal solution of (32) on a restricted domain ($u, t \in \mathbb{R}_+, u < t, x \in X$, and q-bounded sets $B \in \mathfrak{B}(X)$).

THEOREM 4.3. *The function $\bar{P}(u, x; t, B)$, being restricted to q-bounded sets B, is the minimal non-negative solution of the forward Kolmogorov equation (32). Also, if $\bar{P}$ is a transition probability function (that is, $\bar{P}(u, x; t, X) = 1$ for all u, x, t in the domain of $\bar{P}$), then $\bar{P}$ is the unique non-negative solution of the forward Kolmogorov equation (32) that is a measure on $(X, \mathfrak{B}(X))$ for fixed u, x, t with $u < t$ and takes values in $[0, 1]$.*

PROOF. Let P^* defined on the same domain as $\bar{P}$ be a non-negative solution of the forward Kolmogorov equation (32). Integrating (32) from u to t and by using the boundary condition (31), for all $x \in X$, $u, t \in \mathbb{R}_+$ with $u < t$, and q-bounded sets $B \in \mathfrak{B}(X)$,

(42)
$$P^*(u, x; t, B) = I\{x \in B\} e^{-\int_u^t q(x, s) ds} + \int_u^t ds \int_X \Pi(s, y; t, B) P^*(u, x; s, dy),$$

for all q-bounded sets B. Since the last term of (42) is non-negative,

$$(43) \qquad P^*(u,x;t,B) \geq I\{x \in B\}e^{-\int_u^t q(x,s)ds} = \bar{P}^{(0)}(u,x;t,B),$$

where the last equality is (14). For all x, u, t with $u < t$ and q-bounded sets B, assume $P^*(u,x;t,B) \geq \sum_{m=0}^{n} \bar{P}^{(m)}(u,x;t,B)$ for some $n \geq 0$. Then from (42)

$$P^*(u,x;t,B) \geq I\{x \in B\}e^{-\int_u^t q(x,s)ds} + \int_u^t ds \int_X \Pi(s,y;t,B) \sum_{m=0}^{n} \bar{P}^{(m)}(u,x;s,dy)$$

$$= \bar{P}^{(0)}(u,x;t,B) + \sum_{m=0}^{n} \bar{P}^{(m+1)}(u,x;t,B) = \sum_{m=0}^{n+1} \bar{P}^{(m)}(u,x;t,B).$$

Thus, by induction, $P^*(u,x;t,B) \geq \sum_{m=0}^{n} \bar{P}^{(m)}(u,x;t,B)$ for all $n \geq 0$, $x \in X$, $u, t \in \mathbb{R}_+$ with $u < t$, and q-bounded sets $B \in \mathfrak{B}(X)$, which implies that $P^*(u,x;t,B) \geq \bar{P}(u,x;t,B)$ for all u, x, t with $u < t$, and for all q-bounded sets B.

To prove the uniqueness property of $\bar{P}$, let the solution P^* be a measure on $(X, \mathfrak{B}(X))$ for fixed u, x, t with $u < t$ and with values in $[0,1]$. It follows from statement (i) of the theorem that for all $B \in \mathfrak{B}(X)$

$$(44) \quad P^*(u,x;t,B) = \lim_{n \to \infty} P^*(u,x;t,B \cap B_n) \geq \lim_{n \to \infty} \bar{P}(u,x;t,B \cap B_n) = \bar{P}(u,x;t,B),$$

where $\{B_n\}_{n \geq 0}$ is an increasing sequence of q-bounded sets such that $B_n \uparrow X$ as $n \uparrow \infty$, whose existence is guaranteed by Assumption 1.1. If $\bar{P}(u,x;t,X) = 1$ for all u, x, t, then the uniqueness of $\bar{P}$ within the set of solutions to the forward Kolmogorov equation that take values in $[0,1]$ and that are measures on $(X, \mathfrak{B}(X))$ for fixed u, x, t with $u < t$ follows from the minimality of $\bar{P}$ (44) and from the same arguments as in the proof of uniqueness in Theorem 3.2.   □

REMARK 4.1.   The results of this paper can be extended to non-conservative Q-functions. As mentioned in section 1, any non-conservative Q-function q can be transformed into a conservative Q-function by adding a state $\tilde{x}$ to X with $q(x,t,\{\tilde{x}\}) := -q(x,t,X)$, $q(\tilde{x},t,X) := 0$, and $q(\tilde{x},t,\{\tilde{x}\}) := 0$, where $x \in X$ and $t \in \mathbb{R}_+$. According to Theorem 2.2, there is a transition function $\bar{P}$ of a jump Markov process with the state space $\tilde{X} = X \cup \{\tilde{x}\}$, and this process is determined by the initial state distribution and by the compensator defined by the modified Q-function. The proofs of the results of sections 3 and 4 do not use the assumption that the Q-function q is conservative. Therefore, these results remain valid for non-conservative Q-functions. However, the validity of the condition $\bar{P}(u,x;t,X) = 1$ for all x, u, t with $u < t$ in Theorems 3.2 and 4.3 is possible only if $q(x,t,X) = 0$ almost everywhere in t for each $x \in X$. Thus, in fact, q is conservative, if $\bar{P}(u,x;t,X) = 1$ for all x,

u, t with $u < t$. It is also easy to see that the minimal solutions of both the backward and forward Kolmogorov equations are equal to $\bar{P}(u, x; t, B)$, when $x \in X$ and $B \in \mathfrak{B}(X)$, where the transition function $\bar{P}$ is described in the previous paragraph for a broader domain.

REMARK 4.2. In this paper, transition functions $P(u, x; t, B)$ and $\bar{P}(u, x; t, B)$ are defined for $u > 0$. All the results of Sections 3 and 4 hold for $u \geq 0$ with the same proofs. When x is the initial state of the process $\mathbb{X}$, the results of Section 2 also hold for $u \geq 0$.

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